

**THE DECAY ESTIMATE AND ASYMPTOTIC BEHAVIOUR OF
THE BLOW UP TIME FOR EVOLUTION EQUATION WITH A
NON LINEAR SOURCE**

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Abstract: In this paper, we consider the following initial-boundary problem

$$(P) \begin{cases} u_{tt}(x, t) - \xi Lu(x, t) + b|u_t(x, t)|^q = f(u(x, t)) & \text{in } \Omega \times (0, T), \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = 0 & \text{in } \Omega, \\ u_t(x, 0) = 0 & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, L is an elliptic operator, where initial data in which our initial energy can take positive values, with initial and boundary conditions of Dirichlet type, and the nonlinear function $f(s)$ is a positive, increasing and convex function for the nonnegative values of s and b, ξ is a positive parameter.

This work is concerned with a nonlinear wave equation with nonlinear source terms acting in this equation. We will prove that the solution of our considered problem blows up in finite time provided that the initial data and the parameter ξ are small enough.

Under some assumptions, we show that the solution of the above problem blows up in a finite time and its blow-up time goes to the one of the solution of the following differential equations

$$\begin{cases} \alpha''(t) = f(\alpha(t)), & t > 0 \\ \alpha(0) = 0, & \alpha'(0) = 0. \end{cases}$$

Finally, we give some numerical results to illustrate our analysis.

Keywords and Phrases: Nonlinear wave equation, nonlinear wave, blow-up, convergence, initial boundary value problem, nonlinear hyperbolic equation, asymptotic behavior, finite difference method, numerical blow-up time.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Consider the following initial-boundary value problem

$$u_{tt}(x, t) - \xi Lu(x, t) + b|u_t(x, t)|^q = f(u(x, t)) \quad \text{in } \Omega \times (0, T), \quad (1.1)$$

$$u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (1.2)$$

$$u(x, 0) = 0 \quad \text{in } \Omega, \quad (1.3)$$

$$u_t(x, 0) = 0 \quad \text{in } \Omega, \quad (1.4)$$

where $f(s)$ is a positive, increasing and convex function for the nonnegative values of s , $\int_0^\infty \frac{ds}{f(s)} < +\infty$, and b, ξ is a positive parameter,

A great number of processes of the applied sciences can be modeled by means of evolution equations involving differential operators, or systems of such equations, hyperbolic partial differential equations are used to describe evolutionary processes with the property that information propagate with a finite speed. One of the simplest and therefore standard models is that of the free wave equation.

$$u_{tt} - c^2 \Delta u = 0,$$

where c denotes the speed of propagation and $\Delta = \sum_{i=1}^n \partial_i^2$, the usual Laplacian in Euclidean space. This equation arises together with certain initial and boundary conditions if one models the oscillatory behaviour of vibrating strings, membranes or the propagation of sound.

Here $u = u(t, x)$ denotes a displacement or a pressure and thus a time-dependent scalar field. In electrodynamics the unknowns are the electric and the magnetic field, which satisfy in vacuum a related equation. Oscillations of vibrating strings

and membranes are described by quasi-linear equations due to a relation between length/area deformation and energy.

Solutions of nonlinear wave equations which blow up in a finite time have been the subject of investigation of many authors (see [5], [8], [12], [14], [28], [29], [35], and the references cited therein).

By standard methods, local existence, uniqueness, blow up and global existence have been treated by a number of authors (Keller [23], Sattinger [35]) where they have shown that the solutions to the initial value problem or to initial-boundary value problems for classical nonlinear wave equations in the form

$$Pu_{tt} + Au = F(u),$$

in one, two or three dimensions are not stable in time for arbitrary initial data and arbitrary nonlinearities. Their proofs of these results are based upon a comparison principle together with a Huygens-Fresnel principle coupled with solving an initial value problem for an associated ordinary differential equation in time. The size of the nonlinearity generally determines the escape time in their proofs. They proved under some restrictions on the parameters and the initial data several results on local existence and global existence of a weak solution. They also showed that any weak solution with negative initial energy blows up in finite time. See in this regard [10, 11], [15], [20], [21], [24], [28], [29-30], [38], [32], [42], and references therein.

We mention also the work by Levine and Todorova [26] in which the authors considered the following Cauchy problem:

$$u_{tt} - \Delta u + a|u_t|^{q-2} - b|u|^{d-2}u = 0,$$

For $b = 0$, it is well known that the damping term $a|u_t|^{q-2}$ assures global existence and decay of the solution energy for arbitrary initial data (see [17]).

For $a = 0$, the source term causes finite-time blow-up of solutions with a large initial data (negative initial energy) (see [3]). The interaction between the damping term $a|u_t|^{q-2}$ and the source term $b|u|^{d-2}u$ makes the problem more interesting. This situation was first considered by Levine [11, 12] in the linear damping case ($q = 2$), where he showed that solutions with negative initial energy blow up in finite time.

Similar results have also been established by Vitillaro in [38] combined the arguments in [17] to extend these results to situations where the damping is nonlinear and the solution has positive initial energy.

In this paper, we are interested in the asymptotic behavior of the blow-up time when $b = 0$, ξ approaches zero and the initial data is small enough also in the case

where Ω is large enough and its size is taken as parameter.

Our work was motivated by the paper of Friedman and Lacey in [9], where they have considered the following initial-boundary value problem

$$\begin{cases} u_t(x, t) = \varepsilon \Delta u(x, t) + f(u(x, t)) & \text{in } \Omega \times (0, T), \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where Δ is the Laplacian, $f(s)$ is a positive, increasing, convex function for the non-negative values of s , and $\int_0^\infty \frac{ds}{f(s)} < \infty$, $u_0(x)$ is a continuous function in Ω . Under some additional conditions on the initial data we have shown that the solution of the above problem blows up in a finite time and its blow-up time tends to the one of the solution $\lambda(t)$ of the following differential equation

$$\lambda'(t) = f(\lambda(t)), \quad \lambda(0) = M, \quad (1.5)$$

as ε goes to zero where $M = \sup_{x \in \Omega} u_0(x)$.

The proof developed in [8] are based on the construction of upper and lower solutions and it is difficult to extend the above result to the problems described in (1.1)-(1.4).

In this paper, we prove a similar result. Precisely, we show that when ξ is small enough, any solution of (1.1)-(1.4) blows up in a finite time and its blow-up time tends to the one of the solution $\alpha(t)$ of the following differential equation below

$$\alpha''(t) = f(\alpha(t)), \quad \alpha(0) = 0, \quad \alpha'(0) = 0. \quad (1.6)$$

We also prove that the above result remains valid if Ω is large enough and its size is taken as parameter. Our paper is written in the following manner. In the next section, under some assumptions, we show that any solution u of (1.1)-(1.4) blows up in a finite time and its blow-up time goes to the one of the solution $\alpha(t)$ of the differential equation defined in (1.6). Finally, in the last section, we give some numerical results to illustrate our analysis.

2. Preliminaries and Notations

Throughout this paper Ω be a bounded open subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. The elliptic operator in divergence form L is defined as follows

$$Lu = \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right).$$

The coefficient $a_{ij}(x) \in C^1(\Omega)$ satisfy the following relation $a_{ij} : \bar{\Omega} \rightarrow \mathbb{R}$, $a_{ij} \in C^1(\bar{\Omega})$, $a_{ij} = a_{ji}$, $1 \leq i, j \leq N$,

$$\sum_{i,j=1}^N a_{ij}(x) \xi_i \xi_j \geq C |\xi|^2 \quad \text{for } \xi \in \mathbb{R}^N, \quad C > 0,$$

Here $(0, T)$ is the maximal time interval of existence of the solution u . The time T may be finite or infinite. When T is infinite, we say that the solution u exists globally. When T is finite, the solution u develops a singularity in a finite time, namely

$$\lim_{t \rightarrow T} \|u(x, t)\|_{\infty} = +\infty,$$

where $\|u(x, t)\|_{\infty} = \sup_{x \in \Omega} |u(x, t)|$. In this last case, we say that the solution u blows up in a finite time and the time T is called the blow-up time of the solution u .

Throughout this paper, Introduce the function $F(s) = \int_0^s f(\sigma) d\sigma$. and we suppose that

$$\int_0^{\infty} \frac{ds}{\sqrt{F(s)}} < +\infty.,$$

3. Blow-up of the Solution for the Problem

In this section, under some assumptions, we show that any solution u of (1.1)-(1.4) blows up in a finite time and its blow-up time goes to the one of the solution of a differential equation defined in (1.6) when ξ tends to zero.

Before starting, let us recall a well known result. Consider the following eigenvalue problem

$$-L\varphi = \lambda\varphi \quad \text{in } \Omega, \tag{3.1}$$

$$\varphi = 0 \quad \text{on } \partial\Omega, \tag{3.2}$$

$$\varphi > 0 \quad \text{in } \Omega. \tag{3.3}$$

The above problem has a solution (φ, λ) with $\lambda > 0$. We can normalize φ so that $\int_{\Omega} \varphi dx = 1$.

Now, let us state our result on the blow-up.

Theorem 3.1. *Let $F(z) = \int_0^z f(s) ds$ and suppose that $\int_0^{\infty} \frac{ds}{\sqrt{F(s)}} < +\infty$. If $\xi <$*

$\lambda \int_0^\infty \frac{ds}{f(s)}$ then the solution u of (1.1)-(1.4) blows up in a finite time and its blow-up time T satisfies the following relation

$$T = T_e(1 + \frac{\xi A}{2}) + o(\xi) \quad \text{as } \xi \rightarrow 0 \quad (3.4)$$

where $T_e = \frac{1}{\sqrt{2}} \int_0^{+\infty} \frac{ds}{\sqrt{F(s)}}$ is the blow-up time of the solution $\alpha(t)$ of the differential equation defined in (1.6).

Proof. Since $(0, T)$ is the maximal time interval on which the solution u exists, our aim is to show that T is finite and satisfies the above inequality. Introduce the function $v(t)$ defined as follows

$$v(t) = \int_{\Omega} \varphi(x) u(x, t) dx \quad \text{for } t \in (0, T).$$

Take the derivative of v in t and use (1.1) to obtain

$$v''(t) = \xi \int_{\Omega} (Lu) \varphi(x) dx + \int_{\Omega} f(u) \varphi dx.$$

Applying Green's formula, we arrive at

$$v''(t) = \xi \int_{\Omega} u L \varphi dx + \int_{\Omega} f(u) \varphi dx.$$

Using (3.1) and Jensen's inequality, we find that

$$v''(t) \geq -\xi v(t) + f(v(t)),$$

which implies that

$$v''(t) \geq f(v(t)) \left(1 - \frac{\xi v(t)}{f(v(t))} \right).$$

We observe that

$$\int_0^\infty \frac{ds}{f(s)} \geq \sup_{t \geq 0} \int_0^t \frac{ds}{f(s)} \geq \sup_{t \geq 0} \frac{t}{f(t)}$$

because $f(s)$ is an increasing function for the nonnegative values of s . We deduce that $v''(t) \geq (1 - \xi A) f(v(t))$, which implies that

$$v'(t) \geq (1 - \xi A) \int_0^t f(v(s)) ds, \quad t \in (0, T). \quad (3.5)$$

$$v(0) = 0, \quad v'(0) = 0. \quad (3.6)$$

Let $\beta(t)$ be the solution of the following differential equation

$$\beta'(t) = (1 - \xi A) \int_0^t f(\beta(s)) ds, \quad t \in (0, T_0). \quad (3.7)$$

$$\beta(0) = 0, \quad \beta'(0) = 0, \quad (3.8)$$

where $(0, T_0)$ is the maximal time interval of existence of $\beta(t)$. It is not hard to see that

$$\beta''(t) = (1 - \xi A) f(\beta(t)).$$

Multiply both sides of the above equality by $\beta'(t)$ to obtain

$$\left(\frac{(\beta'(t))^2}{2} \right)' = (1 - \xi A) (F(\beta(t)))_t. \quad (3.9)$$

Integrating the equality in (3.9) over $(0, t)$, we find that

$$\frac{(\beta'(t))^2}{2} = (1 - \xi A) (F(\beta(t))), \quad (3.10)$$

which implies that

$$\beta'(t) = \sqrt{2(1 - \xi A) F(\beta(t))}.$$

This equality may be rewritten as follows

$$\frac{d\beta}{\sqrt{F(\beta)}} = \sqrt{(1 - \xi A)} dt.$$

After integration over $(0, T_0)$, we discover that

$$T_0 = \frac{1}{\sqrt{2(1 - \xi A)}} \int_0^\infty \frac{ds}{\sqrt{F(s)}}.$$

Since the above integral converges, we see that $\beta(t)$ blows up at the time T_0 . On the other hand, the maximum principle implies that

$$v(t) \geq \beta(t) \quad \text{for } t \in (0, T_*), \quad (3.11)$$

where $T_* = \min\{T_0, T\}$. We deduce that $T \leq T_0$. Indeed, suppose that $T > T_0$. From (3.11), it is not difficult to see that $v(T_0) = \infty$ which implies that u blows

up at the times T_0 . But this contradicts the fact that $(0, T)$ is the maximal time interval of existence of the solution u . Hence, we have

$$T \leq T_0 = \frac{1}{\sqrt{(1-\xi A)}} \int_0^\infty \frac{ds}{\sqrt{F(s)}}. \quad (3.12)$$

Now let us define the function $U(t)$ as follows

$$U(t) = \sup_{x \in \Omega} u(x, t) \quad \text{for } t \in (0, T).$$

Obviously, we have $U(0) = 0$ and $U'(0) = 0$ and there exists $x_0 \in \Omega$ such $U(t) = u(x_0, t)$. It is not hard to see that $Lu(x_0, t) \leq 0$. Consequently, we get

$$\begin{cases} U''(t) \leq f(U(t)), \\ U(0) = 0, \quad U'(0) = 0, \end{cases}$$

which implies that

$$U'(t) \leq \int_0^t f(U(s)) ds, \quad (3.13)$$

$$U(0) = 0, \quad U'(0) = 0. \quad (3.14)$$

Let ψ be the solution of the differential equation below

$$\psi'(t) = \int_0^t f(\psi(s)) ds, \quad t \in (0, T_1), \quad (3.15)$$

$$\psi(0) = 0, \quad \psi'(0) = 0, \quad (3.16)$$

where $(0, T_1)$ is the maximal time interval of existence of $\psi(t)$. As we have seen for the solution $\beta(t)$, $\psi(t)$ blows up at the time $T_1 = \frac{1}{\sqrt{2}} \int_0^\infty \frac{ds}{\sqrt{F(s)}}$. By the maximum principle, we find that

$$U(t) \leq \psi(t) \quad \text{for } t \in (0, T_{**}),$$

where $T_{**} = \min\{T, T_1\}$. This implies that $T_{**} = T$. In fact, if $T_1 > T$, we obtain $U(T) \leq \psi(T) < +\infty$ which is a contradiction. Therefore

$$T \geq T_1 = \frac{1}{\sqrt{2}} \int_0^\infty \frac{ds}{\sqrt{F(s)}}. \quad (3.17)$$

Apply Taylor's expansion to obtain

$$\frac{1}{\sqrt{(1-\xi A)}} = 1 + \frac{\xi A}{2} + o(\xi).$$

Use (3.12), (3.17) and the above relation to complete the rest of the proof. $\square$

Remark 3.1. If $f(s) = e^s$ then $F(s) = e^s - 1$. Therefore $T_e = \frac{1}{\sqrt{2}} \int_0^{+\infty} \frac{ds}{\sqrt{e^s - 1}}$ and its value is slightly equal 2.22.

4. Numerical Results

In this section, we consider the radial symmetric solutions of (1.1)-(1.4) when $\Omega = B(0, 1)$, $L = \Delta$ and $f(u) = e^u$. Hence the problem (1.1)-(1.4) may be rewritten as follows

$$u_{tt} = \xi \left(u_{rr} + \frac{N-1}{r} u_r \right) + f(u), \quad r \in (0, 1), \quad t \in (0, T) \quad (4.1)$$

$$u_r(0, t) = 0, \quad u(1, t) = 0 \quad t \in (0, T), \quad (4.2)$$

$$u(r, 0) = u_0 \quad u_t(r, 0) = 0, \quad r \in (0, 1). \quad (4.3)$$

Let I be a positive integer and let $h = 1/I$. Define the grid $x_i = ih$, $0 \leq i \leq I$ and approximate the solution u of (4.1)-(4.3) by the solution $U_h^{(n)} = (U_0^{(n)}, \dots, U_I^{(n)})$ of the following explicit scheme

$$\begin{aligned} \frac{U_0^{(n+1)} - 2U_0^{(n)} + U_0^{(n-1)}}{\Delta t_n^2} &= \xi N \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} + f(U_0^{(n)}), \\ \frac{U_i^{(n+1)} - 2U_i^{(n)} + U_i^{(n-1)}}{\Delta t_n^2} &= \xi \left(\frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + \frac{(N-1)}{ih} \frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2h} \right) \\ &\quad + f(U_i^{(n)}) \quad 1 \leq i \leq I-1, \\ U_I^{(n)} &= 0, \\ U_i^{(0)} &= 0, \quad U_i^{(1)} = 0, \quad 0 \leq i \leq I \end{aligned}$$

Here, we take $\Delta t_n = \min(\frac{h^2}{2}, f(\|U_h^n\|_\infty^{-\frac{1}{2}}))$ with $\|U_h^n\|_\infty = \sup_{0 \leq i \leq I} |U_i^n|$. Let us notice that the condition $\Delta t_n \leq \frac{h^2}{2}$ ensures the stability of the explicit scheme. We also approximate the solution u of (4.1) – (4.3) by the solution $U_h^{(n)}$ of the implicit

scheme below

$$\begin{aligned}\frac{U_0^{(n+1)} - 2U_0^{(n)} + U_0^{(n-1)}}{\Delta t_n^2} &= \xi N \frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{h^2} + f(U_0^{(n)}) \quad n > 0, \\ \frac{U_i^{(n+1)} - 2U_i^{(n)} + U_i^{(n-1)}}{\Delta t_n^2} &= \xi \left(\frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} + \frac{(N-1)}{ih} \frac{U_{i+1}^{(n+1)} - U_{i-1}^{(n+1)}}{2h} \right) \\ &\quad + f(U_i^{(n)}), \quad 1 \leq i \leq I-1, \\ U_I^{(n+1)} &= 0 \quad n > 0, \\ U_i^{(0)} &= 0, \quad U_i^{(1)} = 0, \quad 0 \leq i \leq I\end{aligned}$$

where $\Delta t_n = \min(\frac{h^2}{2}, f(\|U_h^n\|_\infty^{-\frac{1}{2}}))$

The explicit scheme may be written as follows:

$$\begin{aligned}\frac{U_i^{(n+1)} - 2U_i^{(n)} + U_i^{(n-1)}}{\Delta t_n^2} &= \xi \left(\frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + \frac{(N-1)}{ih} \frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2h} \right) \\ &\quad + f(U_i^{(n)}), \quad 1 \leq i \leq I-1, \\ U_i^{(n+1)} - 2U_i^{(n)} + U_i^{(n-1)} &= \Delta t_n^2 \xi \left(\frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + \frac{(N-1)}{ih} \frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2h} \right) \\ &\quad + \Delta t_n^2 f(U_i^{(n)}), \quad 1 \leq i \leq I-1,\end{aligned}$$

or

$$\begin{aligned}U_i^{(n+1)} &= 2U_i^{(n)} - U_i^{(n-1)} + \Delta t_n^2 \xi \left(\frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + \frac{(N-1)}{ih} \frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2h} \right) \\ &\quad + \Delta t_n^2 f(U_i^{(n)}), \quad 1 \leq i \leq I-1,\end{aligned}$$

or

$$\begin{aligned}U_i^{(n+1)} &= 2U_i^{(n)} - U_i^{(n-1)} + \frac{\xi \Delta t_n^2}{h^2} U_{i+1}^{(n)} - \frac{2\xi \Delta t_n^2}{h^2} U_i^{(n)} + \frac{\xi \Delta t_n^2}{h^2} U_{i-1}^{(n)} \\ &\quad + \frac{\xi(N-1)\Delta t_n^2}{2ih^2} U_{i+1}^{(n)} - \frac{\xi(N-1)\Delta t_n^2}{2ih^2} U_{i-1}^{(n)} + \Delta t_n^2 f(U_i^{(n)}), \quad 1 \leq i \leq I-1,\end{aligned}$$

or

$$\begin{aligned}
U_i^{(n+1)} &= \frac{\xi \Delta t_n^2}{h^2} \left(1 + \frac{(N-1)}{2i}\right) U_{i+1}^{(n)} + 2 \left(1 - \frac{\xi \Delta t_n^2}{h^2}\right) U_i^{(n)} \\
&\quad + \frac{\xi \Delta t_n^2}{h^2} \left(1 - \frac{(N-1)}{2i}\right) U_{i-1}^{(n)} - U_i^{(n-1)} + \Delta t_n^2 f(U_i^{(n)}), \quad 1 \leq i \leq I-1, \\
U_I^{(n+1)} &= 0 \quad n > 0, \\
U_i^{(0)} &= 0, \quad U_i^{(1)} = 0, \quad 0 \leq i \leq I
\end{aligned}$$

For $i = 1$, $U_1^{(n+1)} = \frac{\xi \Delta t_n^2}{h^2} \left(1 + \frac{(N-1)}{2}\right) U_2^{(n)} + 2 \left(1 - \frac{\xi \Delta t_n^2}{h^2}\right) U_1^{(n)} + \frac{\xi \Delta t_n^2}{h^2} \left(1 - \frac{(N-1)}{2}\right) U_0^{(n)} - U_1^{(n-1)} + \Delta t_n^2 f(U_1^{(n)})$

or $U_0^{(n)} = 0$ then

For $i = 1$, $U_1^{(n+1)} = \left(2 \left(1 - \frac{\xi \Delta t_n^2}{h^2}\right)\right) U_1^{(n)} + \left(\frac{\xi \Delta t_n^2}{h^2} \left(1 + \frac{(N-1)}{2}\right)\right) U_2^{(n)} - U_1^{(n-1)} + \Delta t_n^2 f(U_1^{(n)})$,

For $i = 2$, $U_2^{(n+1)} = \left(\frac{\xi \Delta t_n^2}{h^2} \left(1 - \frac{(N-1)}{2i}\right)\right) U_1^{(n)} + \left(2 \left(1 - \frac{\xi \Delta t_n^2}{h^2}\right)\right) U_2^{(n)} + \left(\frac{\xi \Delta t_n^2}{h^2} \left(1 + \frac{(N-1)}{2}\right)\right) U_3^{(n)} - U_2^{(n-1)} + \Delta t_n^2 f(U_2^{(n)})$,

For $i = 3$, $U_3^{(n+1)} = \left(\frac{\xi \Delta t_n^2}{h^2} \left(1 - \frac{(N-1)}{2}\right)\right) U_2^{(n)} + \left(2 \left(1 - \frac{\xi \Delta t_n^2}{h^2}\right)\right) U_3^{(n)} + \left(\frac{\xi \Delta t_n^2}{h^2} \left(1 + \frac{(N-1)}{2}\right)\right) U_4^{(n)} - U_3^{(n-1)} + \Delta t_n^2 f(U_3^{(n)})$,

...

For $i = I-1$, $U_{I-1}^{(n+1)} = \left(\frac{\xi \Delta t_n^2}{h^2} \left(1 - \frac{(N-1)}{2i}\right) U_{I-2}^{(n)}\right) + \left(2 \left(1 - \frac{\xi \Delta t_n^2}{h^2}\right)\right) U_{I-1}^{(n)} + \left(\frac{\xi \Delta t_n^2}{h^2} \left(1 + \frac{(N-1)}{2i}\right)\right) U_I^{(n)} - U_{I-1}^{(n-1)} + \Delta t_n^2 f(U_{I-1}^{(n)})$,

or $U_I^{(n)} = 0$ then

For $i = I-1$, $U_{I-1}^{(n+1)} = \left(\frac{\xi \Delta t_n^2}{h^2} \left(1 - \frac{(N-1)}{2i}\right)\right) U_{I-2}^{(n)} + \left(2 \left(1 - \frac{\xi \Delta t_n^2}{h^2}\right)\right) U_{I-1}^{(n)} - U_{I-1}^{(n-1)} + \Delta t_n^2 f(U_{I-1}^{(n)})$,

lead us to the linear system below

$$U_i^{(n+1)} = A U_i^{(n)} + (F^{(n)})_i$$

where A is a $I \times I$ tridiagonal matrix defined as follows

$$A = \begin{pmatrix}
2 \left(1 - \frac{\xi \Delta t_n^2}{h^2}\right) & \frac{\xi \Delta t_n^2}{h^2} \left(1 + \frac{(N-1)}{2i}\right) & 0 & \cdots & 0 \\
\frac{\xi \Delta t_n^2}{h^2} \left(1 - \frac{(N-1)}{2i}\right) & 2 \left(1 - \frac{\xi \Delta t_n^2}{h^2}\right) & \frac{\xi \Delta t_n^2}{h^2} \left(1 + \frac{(N-1)}{2i}\right) & 0 & \cdots \\
0 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \frac{\xi \Delta t_n^2}{h^2} \left(1 + \frac{(N-1)}{2i}\right) \\
0 & 0 & \cdots & \frac{\xi \Delta t_n^2}{h^2} \left(1 - \frac{(N-1)}{2i}\right) & 2 \left(1 - \frac{\xi \Delta t_n^2}{h^2}\right)
\end{pmatrix}$$

implies that

$$A = \begin{pmatrix} a_0 & b_0 & 0 & \cdots & 0 \\ c_0 & a_0 & b_0 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & b_0 \\ 0 & 0 & \cdots & c_0 & a_0 \end{pmatrix}$$

with

$$\begin{aligned} a_0 &= 2\left(1 - \frac{\xi \Delta t_n^2}{h^2}\right), \\ b_0 &= \frac{\xi \Delta t_n^2}{h^2} \left(1 + \frac{(N-1)}{2i}\right), \quad i = 1, \dots, I-2, \\ c_0 &= \frac{\xi \Delta t_n^2}{h^2} \left(1 - \frac{(N-1)}{2i}\right), \quad i = 1, \dots, I-1, \end{aligned}$$

$(F^{(n)})_i = -U_i^{(n-1)} + \Delta t_n^2 f(U_i^{(n)})$ and A a three-diagonal matrix verifying the following properties:

$$A_{i,i} = 2\left(1 - \frac{\xi \Delta t_n^2}{h^2}\right) > 0, \quad 0 \leq i \leq I \text{ and } A_{i-1,i} = \frac{\xi \Delta t_n^2}{h^2} \left(1 - \frac{(N-1)}{2i}\right)$$

$$A_{i,i+1} = \frac{\xi \Delta t_n^2}{h^2} \left(1 + \frac{(N-1)}{2i}\right), \quad 2 \leq i \leq I-2 \text{ so that } A_{i,i} \geq \sum_{i \neq j} A_{i,j}$$

It follows that $U_h^{(n)}$ exists for $n \geq 0$. In addition, since $U_h^{(0)}$ is nonnegative, $U_h^{(n)}$ is also nonnegative for $n \geq 0$.

We need the following definition.

Definition 4.1. We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme blows up in a finite time if $\lim_{n \rightarrow \infty} \|U_h^{(n)}\|_\infty = \infty$ and the series $\sum_{n=0}^\infty \Delta t_n$ converges where $\|U_h^{(n)}\|_\infty = \sup_{0 \leq i \leq I} |U_i^{(n)}|$. The quantity $\sum_{n=0}^\infty \Delta t_n$ is called the numerical blow-up time of the discrete solution $U_h^{(n)}$.

In the following tables, in rows, we present the numerical blow-up times, the numbers of iterations n , the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the The numerical blow-up time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when $\Delta t_n = |T^{n+1} - T^n| \leq 10^{-16}$. The order(s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Numerical experiments for $N = 2$, $f(u) = e^{u(x,t)}$, and $u(x, 0) = 0$

First case: $\xi = 1/5$, $u(x, 0) = 0$

Table 1: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	$CPU\ time$	s
16	2.22401460487680	216	0.015	-
32	2.22219042903374	824	0.031	-
64	2.22164923986443	3152	0.156	1.994
128	2.22149404152201	12037	5.296	1.998
256	2.22143853365078	45874	112.062	1.999
512	2.22143843938750	174408	6115.546	2.000

Table 2: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

I	T^n	n	$CPU\ time$	s
16	2.22401442029031	216	0.015	-
32	2.22190444202100	824	0.031	-
64	2.22164931023410	3152	0.156	1.994
128	2.22149500152201	12037	5.296	1.998
256	2.22143853365245	45874	112.062	1.999
512	2.22143828332936	174408	6115.546	2.000

Numerical experiments for $N = 2$, $f(u) = e^{u(x,t)}$, and $u(x, 0) = 0$

First case: $\xi = 1/10$, $u(x, 0) = 0$

Table 3: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	$CPU\ time$	s
16	2.22062533620715	216	0.015	-
32	2.22123973166955	824	0.031	-
64	2.22139136112212	3152	0.156	1.994
128	2.22142804425813	12037	5.296	1.998
256	2.22143871226464	45874	112.062	1.999
512	2.22143842833291	174408	6115.546	2.000

Table 4: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

I	T^n	n	$CPU\ time$	s
16	2.22062555029006	216	0.015	-
32	2.22124000255652	824	0.031	-
64	2.22139001163220	3152	0.156	1.994
128	2.22142811005221	12037	5.296	1.998
256	2.22143871226464	45874	112.062	1.999
512	2.22143843628366	174408	6115.546	2.000

Numerical experiments for $N = 2$, $f(u) = e^{u(x,t)}$, and $u(x, 0) = 0$

First case: $\xi = 1/25$, $u(x, 0) = 0$

Table 5: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	$CPU\ time$	s
16	2.22062331088738	216	0.015	-
32	2.22123973065225	824	0.031	-
64	2.22139126881106	3152	0.156	1.994
128	2.22142742555813	12037	5.296	1.998
256	2.22143843628332	45874	112.062	1.999
512	2.22143843628329	174408	6115.546	2.000

Table 6: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

I	T^n	n	$CPU\ time$	s
16	2.22062211596601	216	0.015	-
32	2.22139610255772	824	0.031	-
64	2.22139122698531	3152	0.156	1.994
128	2.22149454456510	12037	5.296	1.998
256	2.22143843628292	45874	112.062	1.999
512	2.22143843629532	174408	6115.546	2.000

Numerical experiments for $N = 2$, $f(u) = e^{u(x,t)}$, and $u(x, 0) = 0$

First case: $\xi = 1/50$, $u(x, 0) = 0$

Table 7: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	$CPU\ time$	s
16	2.22062315691215	216	0.015	-
32	2.22124002266589	824	0.031	-
64	2.22139156335584	3152	0.156	1.994
128	2.22149454456510	12037	5.296	1.998
256	2.22143836331233	45874	112.062	1.999
512	2.22143890662580	174408	6115.546	2.000

Table 8: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

I	T^n	n	$CPU\ time$	s
16	2.22062321224253	216	0.015	-
32	2.22124055336621	824	0.031	-
64	2.22139111235889	3152	0.156	1.994
128	2.22142842522332	12037	5.296	1.998
256	2.22143883398787	46874	112.062	1.999
512	2.22143800589660	175408	6115.546	2.000

Numerical experiments for $N = 3$, $f(u) = e^{u(x,t)}$, and $u(x, 0) = 0$

First case: $\xi = 1/100$, $u(x, 0) = 0$

Table 9: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	$CPU\ time$	s
16	2.22062331088015	1143	0.011	-
32	2.22123973086610	4555	0.032	-
64	2.22139126765628	18203	0.156	1.994
128	2.22142867486657	72797	5.296	1.998
256	2.22143799525625	144978	112.062	1.999
512	2.22143842833295	279201	6115.546	2.000

Table 10: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

I	T^n	n	$CPU\ time$	s
16	2.22062325088010	1144	0.011	-
32	2.22123974086616	4556	0.033	-
64	2.22139126786563	18203	0.156	1.994
128	2.22142867486659	72797	5.296	1.998
256	2.22143836398753	145874	112.062	1.999
512	2.22143829065800	274408	6115.546	2.000

Remark 4.1. In the case where the initial data $u(x, 0) = 0$, and the reaction term increases as a function of $f(u) = e^u$, we know that the blow up time of the continuous solution of (1.1)–(1.3) is equal 2.22.

In the case where ξ decays to zero the blow-up time of the solution u is the same as the one of the solution $\alpha(t)$ of the following differential equation

$$\alpha''(t) = f(\alpha(t)), \quad \alpha(0) = 0, \quad \alpha'(0) = 0.$$

We observe also that the value of the blow-up time of $\alpha(t) = 2.22$, see Remark 2.1. We observe that the the blow-up time of the solution is equal 2.22.

We observe from Tables 1–10 that when ξ decays to zero, then the numerical blow-up time of the discrete solution goes to 2.22.

In the following, we give some plots to illustrate our theoretical analysis given in the previous section.

In following Figures 1-16, we can appreciate that the solution blow up in a finite time.

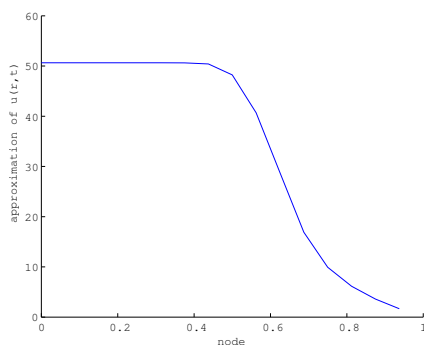


Figure 1: Evolution of discrete solution, source $N = 2$, $f(u) = e^u$, $I = 16$, $\varepsilon = 1/5$, (explicit scheme)

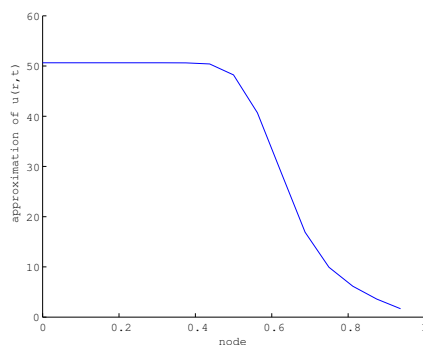


Figure 2: Evolution of discrete solution, source $N = 2$, $f(u) = e^u$, $I = 16$, $\varepsilon = 1/5$, (implicit scheme)

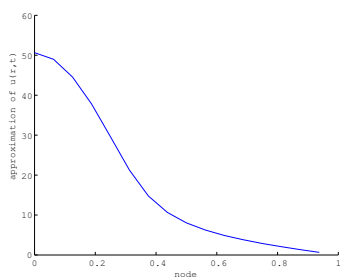


Figure 3: Evolution of discrete solution, source $N = 2$, $f(u) = e^u$, $I = 32$, $\varepsilon = 1/5$, (explicit scheme)

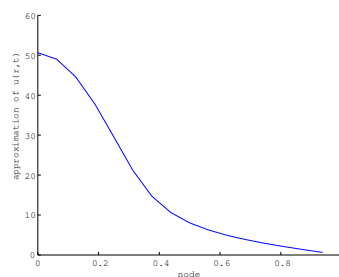


Figure 4: Evolution of discrete solution, source $N = 2$, $f(u) = e^u$, $I = 32$, $\varepsilon = 1/5$, (implicit scheme)

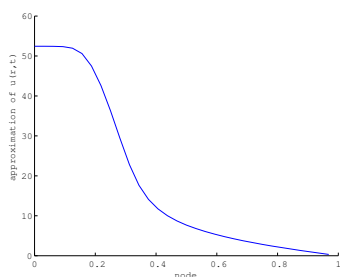


Figure 5: Evolution of discrete solution, source $N = 2$, $f(u) = e^u$, $I = 16$, $\varepsilon = 1/10$, (explicit scheme)

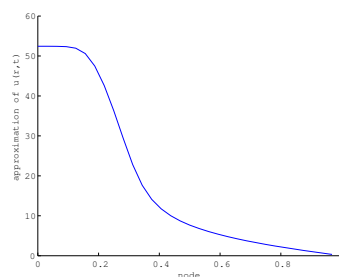


Figure 6: Evolution of discrete solution, source $N = 2$, $f(u) = e^u$, $I = 16$, $\varepsilon = 1/10$, (implicit scheme)

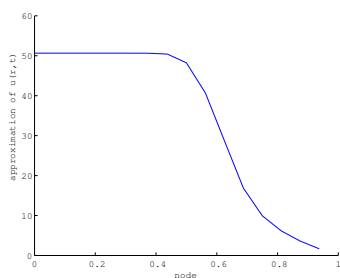


Figure 7: Evolution of discrete solution, source $N = 2$, $f(u) = e^u$, $I = 32$, $\varepsilon = 1/10$, (explicit scheme)

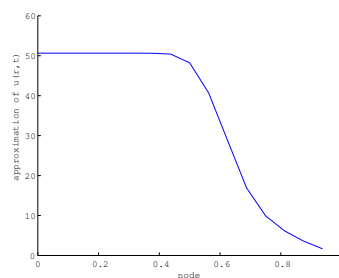


Figure 8: Evolution of discrete solution, source $N = 2$, $f(u) = e^u$, $I = 32$, $\varepsilon = 1/10$, (implicit scheme)

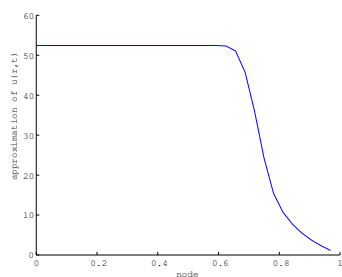


Figure 9: Evolution of discrete solution, source $N = 3$, $f(u) = e^u$, $I = 16$, $\varepsilon = 1/100$, (explicit scheme)

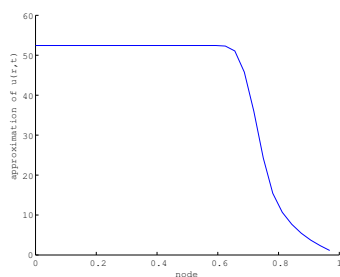


Figure 10: Evolution of discrete solution, source $N = 3$, $f(u) = e^u$, $I = 16$, $\varepsilon = 1/100$, (implicit scheme)

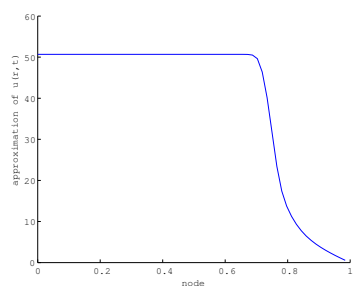


Figure 11: Evolution of discrete solution, source $N = 3$, $f(u) = e^u$, $I = 64$, $\varepsilon = 1/100$, (explicit scheme)

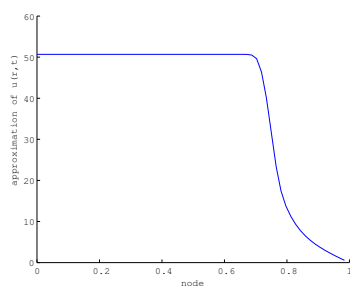


Figure 12: Evolution of discrete solution, source $N = 3$, $f(u) = e^u$, $I = 64$, $\varepsilon = 1/100$, (implicit scheme)

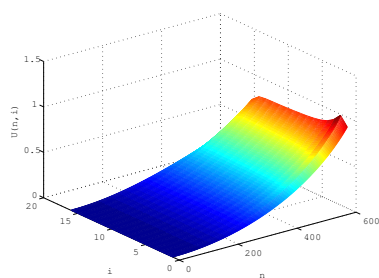


Figure 13: Evolution of discrete solution, source $N = 2$, $f(u) = e^u$, $I = 16$, $\varepsilon = 1/50$, (explicit scheme)

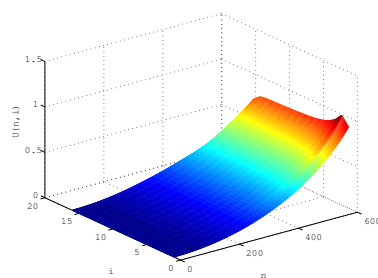


Figure 14: Evolution of discrete solution, source $N = 2$, $f(u) = e^u$, $I = 16$, $\varepsilon = 1/50$, (implicit scheme)

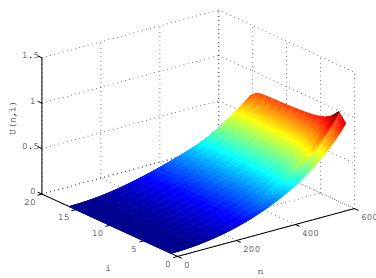


Figure 15: Evolution of discrete solution, source $N = 2$, $f(u) = e^u$, $I = 32$, $\varepsilon = 1/50$, (explicit scheme)

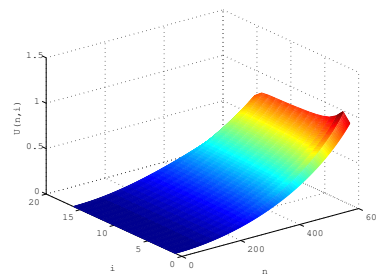


Figure 16: Evolution of discrete solution, source $N = 2$, $f(u) = e^u$, $I = 32$, $\varepsilon = 1/50$, (implicit scheme)

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References

- [1] Abia L. M., López-Marcos J. C. and Martínez J., On the blow-up time convergence of semidiscretizations of reaction-diffusion equations, *Appl. Numer. Math.*, 26 (4) (1998), 399-414.
DOI: [https://doi.org/10.1016/S0168-9274\(97\)00105-0](https://doi.org/10.1016/S0168-9274(97)00105-0)
- [2] Abia L. M., López-Marcos J. C. and Martínez J., Blow-up for semidiscretizations of reaction-diffusion equations, *Appl. Numer. Math.*, 20 (1-2) (1996), 145-156.
DOI: [https://doi.org/10.1016/0168-9274\(95\)00122-0](https://doi.org/10.1016/0168-9274(95)00122-0)
- [3] Ball J., Remarks on blow up and nonexistence theorems for nonlinear evolution equations, *Quart. J. Math. Oxford.*, 28 (1977), 473-486.
- [4] Boni T. K., Extinction for discretizations of some semilinear parabolic equations, *C.R.A.S. Serie I*, 333 (2001), 795-800.
DOI: [https://doi.org/10.1016/S0764-4442\(01\)02078-X](https://doi.org/10.1016/S0764-4442(01)02078-X)
- [5] Boni T. K., Sur l'explosion et le comportement asymptotique de la solution d'une équation parabolique semi-linéaire du second ordre, *C.R.A.S, Serie I*, 326 (3) (1998), 317-322.
DOI: [https://doi.org/10.1016/S0764-4442\(97\)82987-4](https://doi.org/10.1016/S0764-4442(97)82987-4)

- [6] Boni T. K., On blow-up and asymptotic behavior of solutions to a nonlinear parabolic equation of second order with nonlinear boundary conditions, *Comment. Math. Univ. Comenian*, 40 (1999), 457-475.
- [7] Brezis H., Cazenave T., Martel Y. and Ramiandrisos, Blow-up for $u_t = u_{xx} + g(u)$ revisited, *Adv. Diff. Eq.*, 1 (1996), 73-90.
- [8] Deng K., Nonexistence of global solutions of a nonlinear hyperbolic system, *Trans. Am. Math. Soc.*, 349 (1997), 1685-1696.
- [9] Friedman A. and Lacey A. A., The blow-up time for solutions of nonlinear heat equations with small diffusion, *Comment. Math. Univ. Comenian*, 40 (1999), 457-475.
- [10] Gazzola F. and Squassina M., Global solutions and finite time blow up for damped semilinear wave equations, *Ann. I. H. Poincaré*, 23 (2006), 185-207.
- [11] Georgiev V. and Todorova G., Existence of a solution of the wave equation with nonlinear damping and source term, *J. Diff. Eq.*, 109 (1994), 295-308.
- [12] Glassey R. T., Blow-up Theorems for nonlinear wave equations, *Math. Z.*, 132 (1973), 183-203.
- [13] Groisman P., Totally discrete explicit and semi-implicit Euler methods for a blow-up problem in several space dimension, *Computing*, 76(3-4) (2006), 325-352.
DOI: <https://doi.org/10.1007/s00607-005-0136-0>.
- [14] Guowang G. and Shubin W., Existence and nonexistence of global solutions for the generalized, IMBq. equation, *Nonl. Anal. TMA.*, 36 (1999), 961-980.
- [15] Halima Nachid, Koffi N'Guessan and L. B. Sobo Blin, Dirichlet Boundary Conditions and A Potential For A Quenching Time Of Some Non-Linear Wave Equations, *Far East Journal of Mathematical Sciences (FJMS)*, 123 Number 1, Mars (2020), pp. 45-71.
DOI: <http://dx.doi.org/10.17654/MS123010045>.
- [16] Halima Nachid, Quenching For Semi Discretizations Of A Semilinear Heat Equation With Potentiel And General Non Linearities, *Revue D'analyse Numerique Et De Theorie De L'approximation*, 2 (2011), 164- 181.

- [17] Haraux A. and Zuazua E., Decay estimates for some semilinear damped hyperbolic problems, *Arch. Rational Mech. Anal.*, 150 (1988), 191-206.
- [18] Halima Nachid, Behavior Of The Numerical Quenching Time With A Potential And General Nonlinearities, *Journal of Mathematical Sciences Advances and Application*, 15 (2012), 81-105.
- [19] Halima Nachid, F. N'Gohisse and N'Guessan Koffi, The Phenomenon Of Quenching For A Reaction-Diffusion System With Non-Linear Boundary Conditions, *Journal of the Indian Math. Soc. JIMS*, 88 (2021), Nos. (12) (2021),
DOI: <https://doi.org/10.18311/jims/2021/26056>
- [20] Halima Nachid, Benjamin Yekre and Yoro Gozo, Simulation of the blow-up and the quenching time for a positive solutions of singular boundary value problems for a nonlinear parabolic systems, *Africa Mathematics Annals (AFMA)*, 8 (2020), pp. 53-70.
- [21] Ikehata R. and Suzuki T., Stable and unstable sets for evolution equations of parabolic and hyperbolic type, *Hiroshima Math. J.*, 26 (1996), 475-491.
- [22] Kaplan S., On the growth of solutions of quasi-linear parabolic equations, *Comm. Pure Appl. Math.*, 16 (1963), 305-330.
- [23] Keller, J. B., On solutions of nonlinear wave equations, *Comm. Pure Appl. Math.*, 10 (1957), 523-530.
- [24] Levine, L. A., Instability and nonexistence of global solutions to nonlinear wave equations of the form $\rho u_{tt} = -Av + F(u)$, *Trans. Am. Math. Soc.*, 192 (1974), 1-21.
- [25] Levine H. A. and Smiley M. W., Abstract wave equations with a singular nonlinear forcing term, *J. Math. Anal. Appl.* 103, (1984), 409-427.
- [26] Levine H. A. and Todorova G., Blow up of solutions of the cauchy problem for a wave equation with nonlinear damping and source terms and positive initial energy, *Proc. Amer. Math. Soc.*, 129 (2001), 793-805.
- [27] Liang C., Li J., Zhang K., On a hyperbolic equation arising in electrostatic MEMS, *J. Diff. Equations*, (2014), Vol. 256, no. 2, pp. 503-530.

- [28] Merle F. and Zaag H., Existence and universality of the blow-up profile for the semilinear wave equation in one space dimension, *J. Funct. Analysis*, 253, (2007), 43-121.
- [29] Levine L. A., Some additional remarks on the nonexistence of global solutions to nonlinear wave equations, *SIAM J. Math. Anal.*, 5 (1974), 138-146.
- [30] Messaoudi S. and Said-Houari B., Global non-existence of solutions of a class of wave equations with non-linear damping and source terms, *Math. Methods Appl. Sci.*, 27 (2004), 1687-1696
- [31] Protter M. H. and Weinberger H. F., Maximum principles in differential equations, Prentice Hall, Englewood Cliffs, NJ, (1967).
- [32] Rammaha M. A., On the quenching of solutions of the wave equation with a nonlinear boundary condition, *J. Reine Angew. Math.*, 407 (1990), 1-18.
- [33] Reed M., Abstract nonlinear wave equations, *Lecture notes in Mathematics* 507, Springer-Verlag Berlin, New-York, (1976).
- [34] Sang A. and Park R., Remarks on quasilinear hyperbolic equations with dynamic boundary conditions, *Math. Nachr.*, 198 (1999), 169-178.
- [35] Sattinger D. H., On global solution of nonlinear hyperbolic equations, *Arch. Rational Mech. Anal.*, 30 (1968), 148-172.
- [36] Smith R. A., On a hyperbolic quenching problem in several dimensions, *SIAM J. Math. Anal.*, 20 (1989), 1081-1094.
- [37] Stuart A. M. and Floater M. S., On the computation of blow-up, *Euro. J. Appl. Math.*, 1(1) (1990), 47-71.
DOI: <https://doi.org/10.1017/S095679250000005X>
- [38] Vitillaro E., Global existence theorems for a class of evolution equations with dissipation, *Arch. Rational Mech. Anal.*, 149 (1999), 155-182.
- [39] Walter W., *Differential-und Integral-Ungleichungen*, Springer, Berlin, (1954).
- [40] Weissler F. B., An L^∞ blow-up estimate for a nonlinear heat equation, *Communications on Pure and Applied Mathematics.*, 38(3) (1985), 291-295.
DOI: <https://doi.org/10.1002/cpa.3160380303>

- [41] Yekre B., Halima Nachid and Yoro Gozo, Numerical Blow-Up Solutions With Respect To Parameters For Reaction Diffusion Equations, Journal of Mathematical Science Advances and Applications (JMSAA), 56 (2019), pp. 1-37.
- [42] Yang Z., Existence and asymptotic behavior of solutions for a class of quasi-linear evolution equations with non-linear damping and source terms, Math. Meth. Appl. Sci., 25 (2002), 795-814.
- [43] Zhang L, Existence, uniqueness and exponential stability of traveling wave solutions of some integral differential equations arising from neuronal networks, J. Differential Equations, 197 (1), (2004), 162-196.
DOI: [https://doi.org/10.1016/S0022-0396\(03\)00170-0](https://doi.org/10.1016/S0022-0396(03)00170-0)

